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The simple connectivity of links of singularities with one-dimensional critical locus

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Abstract

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A Morse 1-singularity is a germ $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$, with a 1-dimensional singular set, $\Sigma(f)$, such that for each $p\in\Sigma(f)\backslash 0$, the transverse singularity of f at p is Morse. We show that when the Morse 1-singularity f is irreducible, the link, K, of the singularity is not simply connected. We calculate $H_q(K)$ in terms of the covering manifold of K. If an additional hypothesis holds, we give a presentation of $\pi_1(K)$ in terms of the covering manifold of K.

Keywords: Nonisolated singularity, covering manifold.

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Let $f\in\mathbb{C}\{z_0,\ldots,z_n\}$ with $f(0)=0$. The singular set of f is the set

$$\Sigma(f)=\left\{p\in f^{-1}(0):\frac{\mathrm{d}f}{\mathrm{d}z_k}(p)=0,\, k=0,\ldots,n\right\}.$$

Assume $0\in\Sigma(f)$. Then the local topology of the singularity at 0 is described by the objects

$$K=S^{2n+1}\cap f^{-1}(0),\qquad \frac{f}{|f|}:S^{2n-1}-K\to S^1,$$

and

$$F=\left(\frac{f}{|f|}\right)^{-1}(1)\subseteq S^{2n+1}-K,$$

where S^{2n+1} is a sphere of sufficiently small radius, centered at $0\in\mathbb{C}^{n+1}$. The two main theorems governing this situation are the Cone Theorem and the Milnor

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Fibration Theorem. A consequence of the first is that the above objects are independent of the radius of S^{2n+1}, if it is sufficiently small. The second states that $f/|f|: S^{2n+1}\backslash K \to S^1$ is the projection of a locally trivial smooth fibre bundle with fibre $F=(f/|f|)^{-1}(1)$. Two of the main results of [3] concern the connectivity of K and F. It is shown in [3, 5.2] that K is $(n-2)$-connected and in [3, 6.5] that when $\Sigma(f)=\{0\}$, F has the homotopy type of a bouquette of μ n-spheres. For the exact statements of these results see [3] or [12]. The above results also hold for analytic maps and germs of analytic maps $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$.

In this paper we investigate the connectivity of the link for germs of analytic functions $f:(\mathbb{C}^3,0)\to\mathbb{C}$ that have a special type of nonisolated singularity at $0\in\mathbb{C}^3$. Siersma has defined in [18] a class of nonisolated singularities called k-isolated singularities, i.e., f is a k-isolated singularity if $\dim_{\mathbb{C}}\Sigma(f)=k$. We say that a 1-isolated singularity is a Morse 1-singularity, or M1S, if at each point $p\in\Sigma(f)\backslash\{0\}$ the transverse singularity at p is Morse. Using the covering manifold, $(\bar{K},\theta)$, of [6], we give $H_q(K)$ in terms of $H_q(\bar{K})$. We show that for f irreducible $\theta_*: H_1(\bar{K})\to H_1(K)$ is never onto so that K is never simply connected. We show that in certain instances $\theta_*:\pi_1(\bar{K})\to\pi_1(K)$ is injective. These parallel a result of Mumford [5] for normal surfaces.

The structure of the complement $S^{2n+1}\backslash K$ for nonisolated singularities has received much consideration by Siersma and others [3, 7-11, 14-18, 20]. The deformation theory of these singularities and finite determinacy questions relating to the study of these singularities have been investigated by Mond in [4], Pellikaan in [7-11] and Siersma in [15, 16].

This work was a natural outgrowth of the work in [19] on isolated line singularities. We thank Richard Randell for his many helpful comments. Thanks are also given to Dirk Siersma for his comments and to the referee for the obvious care that was taken reading the original manuscript. His/her comments were most appreciated. The shorter and more topological, proof of Lemma 3.2 that he/she pointed out is especially nice.

1. Definitions

Definition 1.1. Let $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ be analytic. f has, or is, a *Morse* 1*-singularity* if $\dim_{\mathbb{C}}\Sigma(f)=1$ and for each $p\in\Sigma(f)\backslash\{0\}$ there are local coordinates (x,y,z) for $\mathbb{C}^3$ about p with $f(x,y,z)=y^2+z^2$. We abbreviate this by saying f is an M1S. There is the obvious extension of the definition of an M1S to germs of analytic functions $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$.

Observe that by [7, 6.3] any germ f that is an M1S is analytically right-equivalent to polynomial. Thus we will usually not distinguish between polynomials, analytic functions, or germs when talking about M1S's. The following result from [6] is stated only for polynomials $f\in\mathbb{C}[x,y,z]$ but, by the preceding, it holds for analytic maps or germs, $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$, with an M1S at 0.

Theorem 1.2. *Let $f \in \mathbb{C}[x, y, z]$, $\dim_{\mathbb{C}} \Sigma(f) = 1$, K the link of the singularity at 0, and $\Sigma = \Sigma(f) \cap K$. Then there is a unique pair $(\bar{K}, \theta)$ with $\bar{K}$ a closed 3-manifold, $\theta : \bar{K} \to K$ continuous, and such that the following hold:*

(1) $\theta : \bar{K} \to K$ is a closed topological immersion onto K with the restriction to $\theta^{-1}(K \backslash \Sigma)$ a homeomorphism onto $K \backslash \Sigma$ and $\theta^{-1}(K \backslash \Sigma)$ dense in $\bar{K}$.

(2) Given any manifold L and closed immersion μ onto K with $\mu^{-1}(K - \Sigma)$ dense in L, there is a unique map $\bar{\mu} : L \to \bar{K}$ such that $\theta \circ \bar{\mu} = \mu$.

$\bar{K}$ is called the covering manifold of K and is used in [6] to analyze the topology of $f^{-1}(0)$ when $f^{-1}(0)$ is invariant under a "good" $\mathbb{C}^*$-action. We do not assume the existence of such an action in this paper.

For the remainder of this paper we restrict ourselves to Morse 1-singularities. Observe that when f is an M1S, $\Sigma = \Sigma(f) \cap K$ is a disjoint union $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ with each Σ_k a circle. For f an M1S and $p \in \Sigma_k$, there are local coordinates (x, y, z) for $\mathbb{C}^3$ about p such that, near p, $f(x, y, z) = y^2 + z^2$. It is, therefore, easy to see that $\theta \,|\, \theta^{-1}(\Sigma_k) : \theta^{-1}(\Sigma_k) \to \Sigma_k$ is a double cover of Σ_k. So, depending on whether $\theta^{-1}(\Sigma_k)$ is connected or not, $\theta^{-1}(\Sigma_k)$ is either a single circle or a disjoint union $S^1 \cup S^1$.

When $\Sigma(f)$ is only the x-axis, Σ is then a single circle and by applying results of [7, 15] it is easy to determine the connectivity of $\theta^{-1}(\Sigma)$: $\theta^{-1}(\Sigma)$ is a single circle if and only if the number of D_∞ points in a generic approximation of f is odd. This number can be calculated by applying results in [15] or more easily by applying [7, 3.18]. The definition of a D_∞ point appears in Section 3 and the generic approximation is defined in [15] and [7, 7.18].

2. The homology of $\bar{K}$ and K

Since we have that $\theta^{-1}(K) = \bar{K}$, we adapt the following notation: for any set $U \subseteq K$ denote the lifting $\theta^{-1}(U)$ to $\bar{K}$ by $\bar{U}$. Thus, $\theta^{-1}(\Sigma) = \bar{\Sigma}$, $\theta^{-1}(\Sigma_k) = \bar{\Sigma}_k$, etc.

Lemma 2.1. *There is a closed neighborhood $\bar{T} \supseteq \mathrm{Int}(\bar{T}) \supseteq \bar{\Sigma}$ such that $\bar{T}$ and $\mathrm{Int}(\bar{T})$ are homotopy equivalent to $\bar{\Sigma}$ and if $\bar{T}_{kj}$ contains a component of $\bar{\Sigma}_k$, then $\bar{T}_{kj}$, and $\mathrm{Int}(\bar{T}_{kj})$ are homotopy equivalent to S^1.*

Proof. The pair $(\bar{K}, \Sigma)$ is triangulable. Take $\bar{T}$ to be a closed PL-regular neighborhood of $\bar{\Sigma}$ and the result follows. □

Lemma 2.2. *There is a closed neighborhood $T \supseteq \mathrm{Int}(T) \supseteq \Sigma$ such that T and $\mathrm{Int}(T)$ are homotopy equivalent to Σ and $\bar{T} = \theta^{-1}(T)$. A similar result holds for each component Σ_k of Σ.*

Proof. Take $T = \theta(\bar{T})$ and the proof is immediate. □

Theorem 2.3. *Suppose the M1S f is analytically irreducible and for each k, $\bar{\Sigma}_k$ is connected. Then $H_3(K) \cong H_3(\bar{K}) \cong Z$, $H_2(K) \cong H_2(\bar{K})$, $H_1(K) \cong H_1(\bar{K}) \oplus (\bigoplus_{k=1}^{n} Z/2Z)$, and $H_0(K) \cong H_0(\bar{K}) \cong Z$.*

Proof. Since f is analytically irreducible, $\bar{K}$ is connected. K is connected by [3]. Let $T = T_1 \cup \cdots \cup T_n$ and $\bar{T} = \bar{T}_1 \cup \cdots \cup \bar{T}_n$ be as in Lemmas 2.2 and 2.1 with $\Sigma_k \subseteq T_k$ and $\bar{\Sigma}_k \subseteq \bar{T}_k$. For each k, $\theta : \bar{\Sigma}_k = S^1 \to \Sigma_k = S^1$ is a double cover. By Theorem 1.3(1), $\theta : \bar{K} \backslash \mathrm{Int}(\bar{T}) \to K \backslash \mathrm{Int}(T)$ is a homeomorphism. Consider the following commutative diagram of long exact homology sequences:

$$\begin{array}{ccccccccc}
H_{q+1}(\bar{K}, \bar{T}) & \longrightarrow & H_q(\bar{T}) & \longrightarrow & H_q(\bar{K}) & \longrightarrow & H_q(\bar{K}, \bar{T}) & \longrightarrow & H_{q-1}(\bar{T}) \\
\downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} \\
H_{q+1}(K, T) & \longrightarrow & H_q(T) & \longrightarrow & H_q(K) & \longrightarrow & H_q(K, T) & \longrightarrow & H_{q-1}(T)
\end{array} \qquad (*)$$

By excision, $H_p(\bar{K}, \bar{T}) \cong H_p(\bar{K} \backslash \mathrm{Int}(\bar{T}), \quad \bar{T} \backslash \mathrm{Int}(\bar{T})) = H_p(\bar{K} \backslash \mathrm{Int}(\bar{T}), \partial \bar{T})$ and $H_p(K, T) \cong H_p(K \backslash \mathrm{Int}(T), \quad K \backslash \mathrm{Int}(T)) = H_p(K \backslash \mathrm{Int}(T), \partial T)$, for all p. Since $\theta : \bar{K} \backslash \bar{\Sigma} \to K \backslash \Sigma$ is a homeomorphism, $\theta_* : H_p(\bar{K} \backslash \mathrm{Int}(\bar{T}), \theta \bar{T}) \to H_p(K \backslash \mathrm{Int}(T), \theta T)$ is an isomorphism for all p. Since $\theta : \bar{\Sigma}_k \to \Sigma_k$ is a connected double cover for all k, and $\bar{T}_k$, T_k have the same homotopy type as $\bar{\Sigma}_k$, Σ_k, respectively, $\theta_* : H_p(\bar{T}_k) \to H_p(T_k)$ is multiplication by ± 2 for $p = 1$, an isomorphism for $p = 0$, and the zero map otherwise. Hence $\theta_* : H_p(\bar{T}) \to H_p(T)$ is an isomorphism for $p = 0$ and $p \geq 2$. It is injective but never onto for $p = 1$. So applying these observations to the diagram we have, by a diagram chase, that $\theta_* : H_q(\bar{K}) \to H_q(\bar{K})$ is an isomorphism for $q \geq 2$ and injective for $q = 1$. $\theta_* : H_0(\bar{K}) \to H_0(K)$ is an isomorphism since both $\bar{K}$ and K are connected. $H_3(\bar{K}) = Z$ since $\bar{K}$ is orientable.

Siersma gave a suggestion leading to the following observation: we may extract the long exact homology sequence below from the preceding diagram by applying a Mayer-Vietoris argument. See [13, 6.6] for the exact definitions of f_j, g_j, h_j.

$$\begin{aligned}
&\to H_2(\bar{T}) \xrightarrow{f} H_2(T) \oplus H_2(\bar{K}) \xrightarrow{g} H_2(K) \xrightarrow{h} H_1(\bar{T}) \\
&\to H_1(T) \oplus H_1(\bar{K}) \to H_1(K) \to .
\end{aligned} \qquad (**)$$

Observe that $H_2(\bar{T}) \cong H_2(T) \cong 0$ since $\bar{T}$ and T have the homotopy type of a disjoint union of circles. Also $H_1(\bar{T}) = \bigoplus_{k=1}^n Z\langle \bar{\alpha}_k \rangle$ and $H_1(T) = \bigoplus_{k=1}^n Z\langle \alpha_k \rangle$ where $\bar{\alpha}_k$ and α_k may be chosen so that $\theta_*(\bar{\alpha}_k) = 2\alpha_k$. Finally, $H_0(\bar{T}) = \bigoplus_{i=1}^n Z$ and $f_0 : H_0(\bar{T}) \to H_0(T) \oplus H_0(\bar{K})$ is injective. Applying these observations, and the facts about the long exact homology sequence of pairs noted above, we may extract the following short exact sequence from (**):

$$0 \to \bigoplus_{k=1}^{n} Z\langle \bar{\alpha}_k \rangle \xrightarrow{f} \left(\bigoplus_{k=1}^{n} Z\langle \alpha_k \rangle \right) \oplus H_1(\bar{K}) \xrightarrow{g} H_1(K) \to 0.$$

Since $g_1 | H_1(\bar{K})$ is injective and $\mathrm{Im} f_1 = \bigoplus_{k=1}^n Z\langle 2\alpha_k \rangle$, it follows that $H_1(K) \cong H_1(\bar{K}) \oplus (\bigoplus_{k=1}^n Z/2Z)$. □

Theorem 2.4. *Suppose the M1S f is irreducible, so that $\bar{K}$ is connected, but for some j, $\bar{\Sigma}_j = \theta^{-1}(\Sigma_j) = \bar{\Sigma}_{j1} \cup \bar{\Sigma}_{j2}$ is not connected. Say $\bar{\Sigma} = (\bigcup_{j=1}^m (\bar{\Sigma}_{j1} \cup \bar{\Sigma}_{j2})) \cup (\bigcup_{k=1}^n \bar{\Sigma}_k^1)$ where $\theta(\bar{\Sigma}_{jk}) = \bar{\Sigma}_j$, $j = 1, \ldots, m$ and $k = 1, 2$. $\theta : \bar{\Sigma}_k^1 \to \Sigma_k^1$ is a connected double cover* *for* $k=1,\ldots,n$. *Then* $H_3(K)\cong H_3(\bar{K})\cong Z$, $H_2(K)\cong H_2(\bar{K})\oplus(\bigoplus_{k=1}^{M} Z)$ *for some* $1\leq M\leq m$, $H_1(K)\cong \theta_*(H_1(\bar{K}))\oplus(\bigoplus_{j=1}^{m} Z)\oplus(\bigoplus_{k=1}^{n} Z/2Z)$, *and* $H_0(K)\cong H_0(\bar{K})\cong Z$.

Proof. Consider diagram (*) in the proof of Theorem 2.3. Excision will again give that $\theta_*: H_q(\bar{K},\bar{T})\to H_q(K,T)$ is an isomorphism. It is then easy to see that $H_3(K)\cong H_3(\bar{K})\cong Z$, that $H_2(\bar{K})$ injects into $H_2(K)$, and that $\theta_*: H_1(\bar{K})\to H_1(K)$ is not surjective. That $\theta_*: H_2(\bar{K})\to H_2(K)$ is not necessarily an isomorphism and $\theta_*: H_1(\bar{K})\to H_1(K)$ is not necessarily injective follows from $\theta^{-1}(\Sigma_j)=\bar{\Sigma}_{j1}\cup\bar{\Sigma}_{j2}$ being a disjoint union of two circles.

Recall that $\bar{T}$ and T have the homotopy types of disjoint unions of circles. They have no second homology. Let $H_1(\bar{T})=(\bigoplus_{j=1}^{m}(Z\langle\bar{\alpha}_{j1}\rangle\oplus Z\langle\bar{\alpha}_{j2}\rangle))\oplus(\bigoplus_{k=1}^{n} Z\langle\bar{\alpha}_k^1\rangle)$ and $H_1(T)=(\bigoplus_{j=1}^{m} Z\langle\alpha_j\rangle)\oplus(\bigoplus_{k=1}^{n} Z\langle\alpha_k^1\rangle)$ where the generators are such that $\theta_*(\bar{\alpha}_{j1})=\theta_*(\bar{\alpha}_{j2})=\alpha_j$ and $\theta_*(\bar{\alpha}_k^1)=2\alpha_k^1$. Applying a Mayer–Vietoris argument to (*) we may again extract the long exact sequence in (**) above. Using the exactness of (**) we extract the short exact sequence below.

$$0\to H_2(\bar{K})\to H_2(K)\to \mathrm{Im}(h_2)\to 0.$$

By definition of f_1, $\ker(f_1)=\mathrm{Im}(h_2)\subset\bigoplus_{j=1}^{m} Z\langle(-\bar{\alpha}_{j1},\bar{\alpha}_{j2})\rangle$. In any case, $\ker(f_1)=\mathrm{Im}(h_2)$ is a free Z-module, $\bigoplus_{j=1}^{M} Z$ with $0\leq M\leq m$. Since $\mathrm{Im}(h_2)$ is free the short exact sequence splits. Therefore, $H_2(K)\cong H_2(\bar{K})\oplus(\bigoplus_{j=1}^{M} Z)$ for some $0<M\leq m$. Note that $\ker(f_0)=\bigoplus Z$ with precisely one Z-summand for each $\bar{\Sigma}_{j1}\cup\bar{\Sigma}_{j2}$. So we may extract the following short exact sequence from (**):

$$0\to \mathrm{Im}(g_1)\to H_1(K)\to \bigoplus_{j=1}^{m} Z\to 0.$$

This sequence splits so $H_1(K)\cong\mathrm{Im}(g_1)\oplus(\bigoplus_{j=1}^{m} Z)$. Further analysis gives the commutative diagram with exact rows shown below:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(f_1) \arrow[r] \arrow[d, leftrightarrow] & H_1(T)\oplus H_1(\bar{K}) \arrow[r] \arrow[d, leftrightarrow] & \mathrm{Im}(g_1) \arrow[r] \arrow[d, leftrightarrow, dashed] & 0 \\
0 \arrow[r] & \mathrm{Im}(f_1) \arrow[r] & H_1(T)\oplus H_1(\bar{K}) \arrow[r] & \theta_*(H_1(\bar{K})\oplus(\bigoplus_{k=1}^{n} Z/2Z) \arrow[r] & 0
\end{tikzcd}$$

A diagram chase shows the dotted arrow can be filled in such a way that the entire diagram commutes. So $\mathrm{Im}(g_1)\cong\theta_*(H_1(\bar{K}))\oplus(\bigoplus_{k=1}^{n} Z/2Z)$. So $H_1(K)$ has the form indicated in the theorem. □

The possibility that $H_1(\bar{K})$ does not inject is illustrated in Example 2.6. A still more detailed analysis of the map $f_1: H_1(\bar{T})\to H_1(T)\oplus H_1(K)$ suggests conditions under which $M=m$. We have the following corollary:

Corollary 2.5. *Let K be the link of an irreducible $M1S$. If $H_1(\bar{K})$ is a finite group, then $H_2(K)=H_2(\bar{K})\oplus(\bigoplus_{j=1}^{m} Z)$ where m is the number of singular circles in K that lift to two circles in $\bar{K}$.*

Example 2.6. Let $\bar{K}$ be the Lens space $L(5,1)$. This is the link of the isolated singularity $x^2+y^2+z^5=0$. Write $\bar{K}=V_1\cup_h V_2$, where the V_k are solid tori and h is a homeomorphism, in the standard way. Let λ_k and μ_k denote the standard longitude and meridian in ∂V_k. Then $h_*([\mu_2])=5[\lambda_1]+[\mu_1]$. Now μ_2 bounds a disk in V_2. Let $\bar{\Sigma}_2$ be a circle contained in the interior of this disk. Let $\bar{\Sigma}_1$ be the center circle of V_1. Let $p:\bar{\Sigma}_2\to\bar{\Sigma}_1$ be a homeomorphism. Let K be the quotient space obtained from $\bar{K}$ by gluing $\bar{\Sigma}_1$ to $\bar{\Sigma}_2$ via p and $\theta:\bar{K}\to K$ the obvious map. The above theorems apply to $\theta:\bar{K}\to K$ and $H_2(K)=Z$, $H_1(K)=Z$. Thus $\theta_*:H_1(\bar{K})\to H_1(K)$ is not injective. The only question is whether K is the link of a Morse 1-singularity.

Corollary 2.7. *Suppose that K is the link of an irreducible $M1S$. Then K is not simply connected. In fact, $H_1(K)\neq 0$.*

Proof. Suppose $\pi_1(K)=1$. Then the map $\theta_*:\pi_1(\bar{K})\to\pi_1(K)$ is surjective. So the induced map $\theta_*:H_1(\bar{K})\to H_1(K)$ is surjective. This contradicts Theorems 2.3 and 2.4. □

3. Morse line singularities: calculating $\pi_1(K)$

This section begins by considering the case of an M1S $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ under the additional assumption that $\Sigma(f)$ is the x-axis, i.e., $\Sigma(f)=\{(x,0,0): x\in\mathbb{C}\}$. Siersma calls such singularities *isolated line singularities* in [15]. We will call them *Morse-line singularities* or MLS's. When $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ is an MLS we can say more about the irreducibility of f and about how K fails to be simply connected.

Definition 3.1. Let f have an MLS at 0. f is *of type A_∞* if and only if there are local coordinates (x,y,z) for $(\mathbb{C}^3,0)$ such that $f(x,y,z)=y^2+z^2$. f is *of type D_∞* if there are local coordinates (x,y,z) such that $f(x,y,z)=xy^2+z^2$.

Lemma 3.2. *Let the germ $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ have an MLS at 0. f factors if and only if f is of type A_∞ if and only if the number of D_∞ points in a generic approximation to f is zero.*

Proof. The equivalence of the second and third conditions is shown by Pellikaan in [7, 7.17, 7.18].

If f is of type A_∞, then $f(x,y,z)=yz$ in some local coordinates for $(\mathbb{C}^3,0)$, y and z are both nonunits.

Assume $f = gh$ for nonunits $g, h \in 0$. The referee pointed out the following proof. By [1], $\dim_z H_1(F) = r - 1$ where F is the Milnor fibre of f and r the number of analytically irreducible components of $f^{-1}(0)$. In this case $r \geq 2$ so that $\dim_z H_1(F) \geq 1$. By [18, 6.2], $H_1(F) = 0$ if and only if the number of D_∞ points in a generic approximation of f is greater than 0. In our situation we have $\#(D_\infty)$ is 0. So, by [7, 7.20, 7.17], f is of type A_∞. $\square$

Now observe that for f an MLS at 0, $\Sigma = \Sigma(f) \cap K$ is a single S^1. For f of type A_∞ it is easy to see that $\bar{K}$ is a disjoint union of two S^3's and K a union of two S^3's along an unknotted circle in each S^3. So for f of type A_∞, $\pi_1(K) = 1$. As an immediate consequence of Theorem 2.5 and Lemma 3.2 we have the following corollary:

Corollary 3.3. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ have an MLS at 0. Then $\pi_1(K) = 1$ if and only if f is of type A_∞.*

It is instructive to compare the situation of an MLS with the situation investigated by Mumford in [5]. He proves that for a $0 \in V$ (V a normal surface) π_1 of the link is trivial if and only if 0 is a simple point of V. Since the singular set of a normal variety has complex codimension at least two, normal *surfaces* have at most isolated singularities. Thus there are no normal surfaces with Morse 1-singularities in $\mathbb{C}^3$. However, in the construction of $(\bar{K}, \theta)$ by Orlik and Wagreich, [6, Section 3], it is observed that $\bar{K}$ is the link of 0 on a normal surface $\bar{V}$ in $\mathbb{C}^N$. As a consequence of Mumford's result and Corollary 3.6 below we observe that if K is the link of an MLS with $\bar{\Sigma}$ connected, then $\pi_1(K) \cong Z/2Z$ if and only if 0 is a simple point of $\bar{V}$.

Also as an immediate consequence of Corollary 3.3 and Lemma 3.2 we get that for f not of type A_∞, $\theta_* : \pi_1(\bar{K}) \to \pi_1(K)$ is never surjective. In fact, more can be said when $\bar{\Sigma}$ is connected.

Lemma 3.4. *Let f be an MLS. Let Σ, $\bar{\Sigma}$, T, and $\bar{T}$ be as above with $\bar{\Sigma}$ connected. Then the following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(\partial\bar{T}) \arrow[rr, "\bar{l}"] \arrow[dl, "\bar{k}"'] \arrow[dd] & & \pi_1(\bar{K}\backslash \mathrm{Int}(\bar{T})) \arrow[dl, "\bar{j}"] \arrow[dd, "\theta_*"] \\
\pi_1(\bar{T}) \arrow[rr, "\bar{i}"] \arrow[dd, "\theta_*"'] & & \pi_1(\bar{K}) \arrow[dd] & \\
 & \pi_1(\partial T) \arrow[rr, "l"] \arrow[dl, "k"'] & & \pi_1(K\backslash \mathrm{Int}(T)) \arrow[dl, "j"] \\
\pi_1(T) \arrow[rr, "i"] & & \pi_1(K) &
\end{tikzcd}$$

where i, j, k, l and $\bar{i}$, $\bar{j}$, $\bar{k}$, $\bar{l}$ are the maps induced by inclusion.

Theorem 3.5. *Let K be the link of the MLS f. Let $(\bar{K}, \theta)$ be its covering manifold. Assume $\bar{\Sigma} = \theta^{-1}(\Sigma)$ is connected. Let $\bar{T}$ be a closed regular neighborhood of $\bar{\Sigma}$, i.e., $\bar{T}$ is a closed 3-torus with $\bar{\Sigma}$ its center circle, and let $T = \theta(\bar{T})$. (This is the set up in Lemma 3.3 and Theorem 3.4.) Then $\theta_*: \pi_1(\bar{K}) \to \pi_1(K)$ is injective but never surjective.*

Proof. It is clear that θ_* is never surjective since the induced map $\theta_*: H_1(\bar{K}) \to H_1(K)$ is never surjective.

Since T is homotopy equivalent to $\Sigma = S^1$ and $\bar{T}$ is homotopy equivalent to $\bar{\Sigma} = S^1$, $\pi_1(T) = Z\langle\beta\rangle$ and $\pi_1(\bar{T}) = Z\langle\alpha\rangle$ where β is the homotopy class of Σ and α the homotopy class of $\bar{\Sigma}$. Furthermore, since $\theta: \bar{\Sigma} \to \Sigma$ is a connected double cover, $\theta_*(\alpha) = \beta^{\pm 2}$. Choose β so that $\theta_*(\alpha) = \beta^2$. Since $\theta: \bar{K}\backslash\bar{\Sigma} \to K\backslash\Sigma$ is a homeomorphism it follows that $\theta_*: \pi_1(\bar{K}\backslash\mathrm{Int}(\bar{T})) \to \pi_1(K\backslash\mathrm{Int}(T))$ and $\theta_*: \pi_1(\partial\bar{T}) \to \pi_1(\partial T)$ are isomorphisms. Note that $\partial\bar{T}$ and ∂T are 2-tori.

Let $\pi_1(\partial T) = \langle\lambda, \mu : \lambda\mu = \mu\lambda\rangle$ and $\pi_1(K\backslash\mathrm{Int}(T)) = \langle g_i : r_j\rangle$. We have $\pi_1(T) = \langle\beta : \_\rangle$. Then by Van Kampen's theorem, and the diagram in Lemma 3.4 we have $\pi_1(K) = \langle g_i, \beta : r_j, l(\mu) = 1, \beta^2 = l(\lambda)\rangle$. Since $\theta_*: \pi_1(\partial\bar{T}) \to \pi_1(\partial T)$ and $\theta_*: \pi_1(\bar{K}\backslash\mathrm{Int}(\bar{T})) \to \pi_1(K\backslash\mathrm{Int}(T))$ are isomorphisms, we may also write $\pi_1(\partial\bar{T}) = \langle\lambda, \mu : \lambda\mu = \mu\lambda\rangle$ and $\pi_1(\bar{K}\backslash\mathrm{Int}(\bar{T})) = \langle g_i : r_j\rangle$. By commutativity of the back and top squares in Lemma 3.4 and Van Kampen's theorem, $\pi_1(\bar{K}) = \langle g_i, \alpha : r_j, \bar{l}(\mu) = 1, \bar{l}(\lambda) = \alpha\rangle$. Since $\bar{l}(\mu)$ and $\bar{l}(\lambda)$ are words in the g_i, we may write $\pi_1(\bar{K}) = \langle g_i : r_j, \bar{l}(\mu) = 1\rangle$. There are two cases to consider: (1) $\bar{l}(\lambda)$ has infinite order in $\pi_1(\bar{K})$ or (2) $\bar{l}(\lambda)$ has finite order.

Suppose $\bar{l}(\lambda)$ has infinite order. In this case take $A = \pi_1(\bar{K}) = \langle g_i : r_j, \bar{l}(\mu)\rangle$, $B = Z = \langle\beta : \_\rangle$, and $G = \pi_1(K) = \langle g_i, \beta : r_j, l(\mu), \beta^2 = l(\lambda)\rangle$. We have $H = Z\langle\bar{l}(\lambda)\rangle \subseteq A$ and $J = Z\langle\beta^2\rangle \subseteq B$. So the assignment $\bar{l}(\lambda) \to \beta^2$ induces an isomorphism between H and J. Therefore by standard results in combinatorial group theory, e.g., [2, 4.3], $A = \pi_1(\bar{K})$ and $B = Z\langle\beta\rangle = \pi_1(T)$ inject into $G = \pi_1(K)$. So the theorem follows in this case.

On the other hand, suppose the order of $\bar{l}(\lambda)$ is n for some $n < \infty$. Note $\bar{l}(\lambda)^n = 1$ is a consequence of $\{r_i\} \cup \{\bar{l}(\lambda)\}$. Take $A = \pi_1(\bar{K}) = \langle g_i : r_j, \bar{l}(\mu)\rangle$ $\langle g_i : r_j, \bar{l}(\mu), \bar{l}(\lambda)^n\rangle$, $B = Z_{2n} = \langle\beta : \beta^{2n} = 1\rangle$, and $G = \langle g_i, \beta : r_j, l(\mu), \beta^2 = l(\lambda)\rangle = \langle g_i, \beta : r_j, l(\mu), l(\lambda)^n, \beta^2 = l(\lambda), \beta^{2n} = 1\rangle$. The second presentation for G follows since $\beta^{2n} = 1$ is a consequence of $\beta^2 = l(\lambda)$ and $l(\lambda)^n = 1$. Finally taking $H = Z_n = \langle\bar{l}(\lambda) : \bar{l}(\lambda)^n = 1\rangle \subseteq A$ and $J = Z_n\langle\beta^2\rangle \subseteq B$, we see that $l(\lambda) \to \beta^2$ induces an isomorphism between H and J. Therefore, by [2, 4.3], both $A = \pi_1(\bar{K})$ and $B = Z_{2n}$ inject into $G = \pi_1(K)$. So the theorem follows in this case. $\square$

Observe that the above proof actually tells how to construct $\pi_1(K)$ from $\pi_1(\bar{K})$: do so by adding a square root of α to $\pi_1(\bar{K})$. Also note that when $\bar{\Sigma}$ is a single S^1, a consequence of the corollary is that a necessary condition for $\bar{K}$ to be an S^3 is $\pi_1(K) = Z_2$.

Corollary 3.6. *Assume K is the link of an MLS and $\bar{\Sigma}$ is a single S^1. Let $\pi_1(\bar{K}) = \langle g_i, \alpha : r_j, \bar{l}(\mu) = 1, \bar{l}(\lambda) = \alpha \rangle$ as above. Then $\pi_1(K) \cong \langle g_i, \alpha, s : r_j, \bar{l}(\mu) = 1, \bar{l}(\lambda) = \alpha, s^2 = \alpha \rangle$.*

Theorem 3.7. *Let K be the link of a Morse 1-singularity and $(\bar{K}, \theta)$ its covering manifold. Assume that for each component circle, Σ_k, of the singular set $\Sigma \subseteq K$, the lifting $\bar{\Sigma}_k$ is connected. As observed above, $\theta : \bar{\Sigma}_k \to \Sigma$ is a double cover. Then $\theta_* : \pi_1(\bar{K}) \to \pi_1(K)$ is injective. More can be said. Let $\bar{T} = \bigcup_{k=1}^n \bar{T}_k$ be a regular neighborhood of $\bar{\Sigma}$ with $\pi_1(\partial \bar{T}_k) = \langle \lambda_k, \mu_k : \lambda_k \mu_k = \mu_k \lambda_k \rangle$ and $\pi_1(\bar{K} \backslash \mathrm{Int}(\bar{T})) = \langle g_i : r_j \rangle$. Let $\pi_1(\bar{\Sigma}_k) = \langle \alpha_k : \_ \rangle$. Then $\pi_1(\bar{K}) = \langle g_i, \alpha_1, \ldots, \alpha_n : r_j, \bar{l}(\mu_k) = 1, \bar{l}(\lambda_k) = \alpha_k, k = 1, \ldots, n \rangle$ and $\pi_1(K) \cong \langle g_i, \alpha_1, \ldots, \alpha_n, s_1, \ldots, s_n : r_j, \bar{l}(\mu_k) = 1, \bar{l}(\lambda_k) = \alpha_k, s_k^2, k = 1, \ldots, n \rangle$.*

Proof. Let $T = \theta(\bar{T})$ and $T_k = \theta(\bar{T}_k)$ as in Theorem 3.5. Define a sequence $\bar{K}_0, \ldots, \bar{K}_k$ as follows:

$$\bar{K}_0 = \bar{K} \backslash \mathrm{Int}(\bar{T}),\ \bar{K}_1 = \bar{K}_0 \cup \bar{T}_1, \ldots, \bar{K}_n = \bar{K}_{n-1} \cup \bar{T}_n = \bar{K}.$$

Define the corresponding sequence for K in the obvious manner. Observe that $\theta : \bar{K}_j \to K_j$, $j = 1, \ldots, n$ is a covering manifold in the same sense as is $\theta : \bar{K} \to K$. The appropriately phrased versions of Theorem 1.2 and Lemma 3.4 hold for $\theta : \bar{K}_j \to K_j$.

Consider $\theta : (\bar{K}_1, \bar{T}_1, \partial \bar{T}_1) \to (K_1, T_1, \partial T_1)$. The proof of Theorem 3.5 applies to this situation since Lemma 3.4 and Theorem 1.2 does. Taking $\pi_1(\bar{K}_0) = \langle g_i : r_j \rangle$ and $\pi_1(\partial \bar{T}_1) = \langle \lambda_1, \mu_1 : \lambda_1 \mu_1 = \mu_1 \lambda_1 \rangle$, and following the proof of Theorem 3.5, we have that $\pi_1(\bar{K}_1) = \langle g_i : r_j, \bar{l}(\mu_1) = 1, \bar{l}(\lambda_1) = 1 \rangle$. An application of [2, 4.3] gives that $\pi_1(K_1) \cong \langle g_i, \alpha_1, s_1 : r_j, \bar{l}(\mu_1) = 1, \bar{l}(\lambda_1) = \alpha_1, s_1^2 = \alpha_1 \rangle$. In particular by taking a little care with base points and an induction it is easy to see that $\pi_1(\bar{K})$ and $\pi_1(K)$ have the form indicated in the theorem. The base point problem is solved by noting that the base point used in ∂T_1 to calculate $\pi_1(K)$ can be pushed to a point in ∂T_2 via a path in K_0 that lifts to a path in $\bar{K}_0$ since $\theta : \bar{K}_0 \to K_0$ is a homeomorphism. $\square$

4. Concluding remarks

There are several questions related to the above results that appear to be of interest. Milnor, in [3], answers the question of when the link of an isolated singularity is a topological sphere, or homology sphere in dimension 3. When dealing with Morse 1-singularities K is not a manifold and the above results show that it is not even a homology 3-sphere. However, $\bar{K}$ could be an S^3. In fact, all the examples in David Mond's list in [4] have $\bar{K} = S^3$. The question of when $\bar{K}$ is an S^3 takes on additional interest in the light of a recent result of Siersma that shows that a second 3-manifold, that in some sense covers K, is never a 3-sphere. His results in [18] shows that the boundary of the Milnor fibre of a Morse 1-singularity is never a sphere.

The general version of the problem mentioned in Example 2.6 also seems to be of interest. It came up in a conversation with Siersma and Randell: Suppose $\bar{K}$ is the link of an isolated algebraic (analytic) singularity. Let $\bar{\Sigma}$ be an algebraic (analytic) link in $\bar{K}$. Let K be the quotient space formed by gluing the circles of $\bar{\Sigma}$ to themselves or to each other in some manner. When is K an algebraic (analytic) link?

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